ON THE STABILITY OF FLOW OF A VISCOUS CONDUCTING FLUID BETWEEN PARALLEL PLANES IN A PERPENDICULAR MAGNETIC FIELD

(OB USTOICHIVOSTI TECHENIIA VIAZKOI PROVODIASHCHEI ZHIDKOSTI MEZHDU PARALLEL'NYMI PLOSKOSTIAMI V PERPENDIKULIARNOM MAGNITNOM POLE)

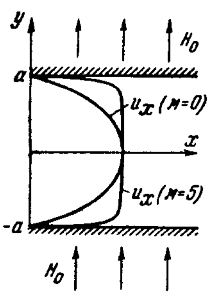
PMM Vol.24, No.4, 1960, pp. 723-725

K.B. PAVLOV and Iu.A. TARASOV
(Moscow)

(Received 15 March 1960)

The stability of flow of a conducting fluid between parallel planes in a magnetic field under the influence of infinitesimally small disturbances has been investigated in a number of papers; in particular, for the case of a longitudinal field, the stability curves have been obtained [1, 2, 3] for various values of the magnetic Reynolds number R_m. In these papers the problem reduces to the determination of the eigenvalues from the solution of a single differential equation of the Orr-Sommerfeld type of the fourth or sixth order, and the corresponding boundary conditions.

If the magnetic field is perpendicular to the flow (Fig. 1), the stationary distributions of the velocity $\mathbf{U}(u_x, 0, 0)$ and the magnetic field $\mathbf{H}(H_x, H_y, 0)$ between the planes will be the following [4]:



$$u_x = wU_0 = U_0 \frac{\cosh M - y\cosh M}{\cosh M - 1} \qquad (M = A(R_m R_g)^{1/2})$$

$$H_x = h\frac{H_0 R_m}{M} = \frac{H_0 R_m}{M}\frac{\sinh My - y\sinh M}{\cosh M - 1} \qquad \left(A = \frac{H}{U_0(4\pi\rho)^{1/2}}\right)$$

$$H_y = H_0 = \mathrm{const}$$

Here H_0 is the uniform external field, U_0 is the velocity at the center of the flow, M is the Hartmann number, R_g is the ordinary Reynolds number, ρ is the density.

In an increasing magnetic field the velocity profile, which is parabolic for $H_0 = M = 0$, deforms in such a way that at large M all the changes in the profile occur in a narrow layer near the wall $\sim a/M$. The stability of such a flow with respect to infinitesimally small disturbances has been investigated [5], under the condition $R_m \ll 1$. The aim of the present work is the analogous problem for values of $R_m \sim 1$, which includes the region of high velocities and temperatures of the order 5000 - 10000°.

If the disturbances to the basic flow (1) are considered to be small, and the system of magnetohydrodynamic equations is linearized in the usual way, then it is possible to find a system of two differential equations for the small disturbances in the y-components of the velocity v_y and the magnetic field h_y. It is convenient to put these in the dimensionless form

$$\frac{R_m}{M} h\Psi - i\frac{\Psi'}{\alpha} = (w - c)\Phi + \frac{i}{\alpha R_m}(\Phi'' - \alpha_1^2\Phi)$$

$$(w - c)(\Psi'' - \alpha_1^2\Psi) - w''\Psi + \frac{i}{\alpha R_g}(\Psi'''' - 2\alpha_1^2\Psi'' + \alpha_1^4\Psi) =$$

$$= \frac{M^2}{R_g R_m}\left\{\frac{R_m}{M} h(\Phi'' - \alpha_1^2\Phi) - \frac{i}{\alpha}(\Phi''' - \alpha_1^2\Phi') - \frac{R_m}{M} h''\Phi\right\} \qquad (2)$$

Here

$$h_y = H_0\Phi(y)\exp\{i[k_x(x - Ct) + k_z z]\}, \qquad \left(c = \frac{C}{U_0}\right)$$
$$v_y = U_0\Psi(y)\exp\{i[k_x(x - Ct) + k_2 z]\},$$

k_x, k_z are free numbers, C is the speed of propagation of the disturbances; primes denote differentiation with respect to $Y = y/a$ (where $2a$ is the distance between the planes), $\alpha = k_x a$, $\alpha_1^2 = a(k_x^2 + k_y^2)$. The boundary conditions for the system (2) have the form

$$\Phi(Y) = \Psi(Y) = \Psi'(Y) = 0 \qquad \text{for } Y = \pm 1 \qquad (3)$$

If it is assumed that $R_m \ll 1$, then the system (2) can be reduced to the Orr-Sommerfeld equation for Ψ

$$(w - c)(\Psi'' - \alpha_1^2\Psi) - w''\Psi + \frac{i}{\alpha R_g}(\Psi'''' - 2\alpha_1^2\Psi'' + \alpha_1^4\Psi) = \qquad (4)$$

which makes it possible to solve the corresponding stability problem by the well-known method of Lin [6] for a much more complicated profile of w. For $R_m \lesssim 1$ the reduction to one equation is not possible, and it is necessary to investigate the system (2) directly. In this connection, use will be made, below, of the usual asymptotic methods [6] to extend the results of [5] to a considerably higher interval of the quantity R_m.

It is known that the magnetic field is a stabilizing factor, i.e. instability occurs at larger values of R_g when a magnetic field is present than when it is absent [1-3, 5]. Therefore, it is natural to look for solutions of (2) in the form of an expansion in powers of $1/\alpha R_g$

$$\Psi,\ \Phi = \Psi^{(0)}, \qquad \Phi^{(0)} + \frac{1}{\alpha R_g}\Psi^{(1)}, \qquad \Phi^{(1)} + \ldots \tag{5}$$

Putting (5) in (2), we have

$$(w-c)(\Psi^{(0)} - \alpha_1^2\Psi) - w''\Psi^{(0)} = 0 \tag{6}$$

$$\frac{R_m}{M}h\Psi^{(0)} - i\frac{\Psi^{(0)\prime}}{\alpha} = (w-c)\Phi^{(0)} + \frac{i}{\alpha R_m}(\Phi^{(0)\prime\prime} - \alpha_1^2\Phi^{(0)}) \tag{7}$$

$$(w-c)(\Psi^{(1)\prime\prime} - \alpha_1\Psi^{(1)}) - w''\Psi^{(1)} + i(\Psi^{(0)\prime\prime\prime\prime} - 2\alpha_1^2\Psi^{(0)\prime\prime} + \alpha_1^4\Psi^{(0)}) =$$

$$= \alpha Mh(\Phi^{(0)\prime\prime} - \alpha_1^2\Phi^{(0)}) - \frac{iM^2}{R_m}(\Phi^{(0)\prime\prime} - \alpha_1^2\Phi^{(0)\prime}) - \alpha Mh'\Phi^{(0)} \tag{8}$$

. . . .

The solutions $\Psi_{1,2} = \Phi_{1,2}^{(0)}$, which may be obtained from (6), are "nonviscous" solutions [6]. A second pair of "nonviscous" solutions $\Phi_{1,2} = \Phi_{1,2}^{(0)}$ is obtained from (7).

Examining Equations (7) and (8), it may be noted that, for $R_m \ll 1$, in the expansion (5) for Ψ, we may restrict ourselves to the zero approximation; however, as R_m increases, the next approximation, generally speaking, may become equal to it, so that it is unjustified to neglect it. Finding the expressions $\Phi^{(0)}$ and $\Psi^{(1)}$ from (7) and (8), it is possible to determine the upper limit on the values of R_m for which it is still possible to restrict oneself to the zero approximation with sufficient accuracy; however, it is simplest to do this with the help of the solution of system (2) in the vicinity of the critical point $Y = Y_c$ (where $w = c$), making use of the fact that with these solutions must be identified both pairs of the "nonviscous" solutions obtained below [6].

Here it is assumed that the flow is at values of $M \sim 1/4$ (for larger values of M it would be necessary to use the approximations introduced in [5]). The influence of the parameter M on the rate of convergence of the expansion (5) is not important; a small increase in M will bring about an appreciable increase in the critical number R_g, at which the flow becomes unstable.

We look for other solutions of system (2) in the form

$$\Psi = \exp\int g\,dY, \qquad \Phi = \exp\int \xi\,dY$$
$$g,\ \xi = (\alpha R_g)^{1/2}g_0,\quad \xi_0 + g_1,\quad \xi_1 + (\alpha R_g)^{-1/2}g_2,\quad \xi_2 + \ldots \tag{9}$$

Equating coefficients of equal powers of aR_g, we find

$$g_0 = \pm\sqrt{i(w-c)}, \qquad g_1 = -\frac{5}{2}\frac{g_0'}{g_0}$$

There is no analogous procedure for finding ξ_0, ξ_1, ... and g_2, g_3, ..., since in each corresponding equation there are two unknowns. If these last terms may be neglected (we shall return to this point below), then there are two solutions

$$\Psi_{3,4} = (w-c)^{-\frac{5}{4}} \exp \mp \int_{Y_c}^{Y} [iaR_g(w-c)]^{1/2} dY] \tag{10}$$

which are "viscous" solutions [6] of the system (2).

The asymptotic expressions (10) are not good in the vicinity of the critical point Y_c. To find the solution in that vicinity, we introduce a new variable $\eta = (Y - Y_c)/\epsilon$, $\epsilon = (aR_g)^{-1/3}$. Putting

$$\Psi, \quad \Phi(Y) = \chi, \qquad \varkappa(\eta) = \chi^{(0)}, \qquad \varkappa + \varepsilon\chi^{(1)}, \qquad \varkappa^{(1)} + \ldots$$

$$w - c = w_c'\varepsilon\eta + w_c''\frac{(\varepsilon\eta)^2}{2!} + \ldots, \qquad w'' = w_c'' + w_c'''\varepsilon\eta + \ldots, \qquad (w_c = w(Y_c) \tag{11}$$

into Equations (2), and equating coefficients of the same powers of ϵ, we obtain

$$i\chi^{(0)''''} + w_c'\eta\chi^{(0)''} = 0, \qquad i\chi^{(1)''''} + w_c'\eta\chi^{(1)''} = w_c''\chi^{(0)} - \frac{1}{2}w_c''\chi^{(0)}\eta^2 \tag{12}$$

$$\varkappa^{(0)''} = 0, \qquad \varkappa^{(1)''} = -R_m\chi^{(0)'}, \ldots \tag{13}$$

Equations (12), (13) give four solutions $\chi = \chi_{1,2,3,4}$, and two solutions $\kappa = \kappa_{1,2}$. If the condition

$$R_m^4 \leqslant R_g \tag{14}$$

is fulfilled, then, for the solutions $\chi = \chi_{1,2,3,4}$, it is sufficiently accurate to restrict the expansion to the first two terms of (11), since under this condition the parameter R_m cannot make the succeeding terms of the expansion equal to these first two. Making use of the asymptotic expansions of the Hankel functions in terms of which the solutions $\chi^{(0)}$ and $\chi^{(1)}$ are expressed, it may be shown that the solutions $\chi = \chi_{1,2}$ are equal to the solutions Ψ_1, Ψ_2, which are found from (6) by the method of Frobenius; the solutions $\chi = \chi_{3,4}$ are equal to Ψ_3, Ψ_4, given by (10); the solutions $\kappa = \kappa_{1,2}$ can be identified with the second pair of nonviscous solutions Φ_1, Φ_2 (in this connection, it is useful to note the analogous results which occur in the case of a longitudinal field [2,3]). Thus it may be concluded that the expansions for the solutions Ψ_1, Ψ_2, Ψ_3, Ψ_4 are selected correctly, so long as condition (14) is fulfilled.

If the above asymptotic methods are applied in looking for solutions of Equations (4), it may be concluded that the corresponding nonviscous solutions, as well as the viscous solutions and the solutions near the critical point, are identical with the solutions $\Psi_{1,2,3,4}(\chi_{1,2,3,4})$ of system (2), to the accuracy possible in the solution of this class of problems. Therefore, if condition (14) is fulfilled, it is possible to investigate Equations (4) together with system (2), which corresponds to the fact that the influence of the magnetic field on the flow stability is due mainly to the change in the basic velocity profile. Since Equations (4) are used to investigate the stability of flows with $R_m \ll 1$, the well-known results for that case [5] may be extrapolated to large values of R_m, up to values of $R_m \sim 1$, so long as condition (14) is still reliably fulfilled. In conclusion, the authors express their thanks to K.P. Staniukovich for discussions of the results.

BIBLIOGRAPHY

1. Stuart, L.T., On the stability of viscous flow between parallel planes in the presence of a coplanar magnetic field. *Proc. Roy. Soc.* A221, No. 1145, p. 189, 1945.

2. Tarasov, Iu.A., Ustoichivost' ploskovo puazeileva techeniia plazmy s konechnoi provodimostiu v magnitnom pole (Stability of plane Poiseuille flow of a plasma with finite conductivity in a magnetic field). *Zh. exsp. teor. fiz.* Vol. 37, No. 6(12), p. 1708, 1959.

3. Velikhov, E.P., Ustoichivost' ploskovo puazeileva techeniia ideal'no provodiashchei zhidkosti v prodol'nom magnitom pole. (Stability of plane Poiseuille flow of an ideal conducting fluid in a longitudinal magnetic field). *Zh. exsp. teor. fiz.* Vol. 36, No. 4, p. 1192, 1959.

4. Landau, L. and Lifshitz, E., *Elektrodinamika sploshnikh sred (Electrodynamics of Continuous Media)*, Section 51. Gostekhteorizdat, 1957.

5. Lock, R.C., The stability of the flow of an electrically conducting fluid between parallel planes under a transverse magnetic field. *Proc. Roy. Soc.* A 233, No. 1192, p. 105, 1955.

6. Lin, C.C., On the stability of two-dimensional parallel flows. *Quart. Appl. Math.* Vol. 3, Nos. 2,3,4, pp. 117-142, 218-234, 277-301, 1945, 1946.

Translated by A.R.